

A Tauberian Theorem for ℓ -adic Sheaves on \mathbb{A}^1

To Wang Yuan on his 80th birthday *

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Abstract

Let $K \in L^1(\mathbb{R})$ and let $f \in L^\infty(\mathbb{R})$ be two functions on \mathbb{R} . The convolution

$$(K * f)(x) = \int_{\mathbb{R}} K(x-y)f(y)dy$$

can be considered as an average of f with weight defined by K . Wiener's Tauberian theorem says that under suitable conditions, if

$$\lim_{x \rightarrow \infty} (K * f)(x) = \lim_{x \rightarrow \infty} (K * A)(x)$$

for some constant A , then

$$\lim_{x \rightarrow \infty} f(x) = A.$$

We prove the following ℓ -adic analogue of this theorem: Suppose K, F, G are perverse ℓ -adic sheaves on the affine line \mathbb{A} over an algebraically closed field of characteristic p ($p \neq \ell$). Under suitable conditions, if

$$(K * F)|_{\eta_\infty} \cong (K * G)|_{\eta_\infty},$$

then

$$F|_{\eta_\infty} \cong G|_{\eta_\infty},$$

where η_∞ is the spectrum of the local field of \mathbb{A} at ∞ .

Key words: Tauberian theorem, ℓ -adic Fourier transformation.

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Introduction

A Tauberian theorem is one in which the asymptotic behavior of a sequence or a function is deduced from the behavior of some of its average. The ℓ -adic Fourier transform was first introduced by Deligne in the study of exponential sums using ℓ -adic cohomology theory. It was further developed by Laumon [5]. In this paper, using the ℓ -adic Fourier transform, we prove an ℓ -adic analogue of Wiener's Tauberian theorem in the classical harmonic analysis. Our study shows that many

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results in the classical harmonic analysis have ℓ -adic analogues and this area has not been fully explored. The result in this paper is absolutely not in its final form.

For any $f_1, f_2 \in L^1(\mathbb{R})$, their convolution $f_1 * f_2 \in L^1(\mathbb{R})$ is defined to be

$$(f_1 * f_2)(x) = \int_{\mathbb{R}} f_1(x-y)f_2(y)dy.$$

If we define the product of two functions to be their convolution, then $L^1(\mathbb{R})$ becomes a Banach algebra. A function $f \in L^\infty(\mathbb{R})$ is called *weakly oscillating* at ∞ if for any $\epsilon > 0$, there exist $N > 0$ and $\delta > 0$ such that for any $x_1, x_2 \in \mathbb{R}$ with the properties that $|x_1|, |x_2| > N$ and $|x_1 - x_2| < \delta$, we have

$$|f(x_1) - f(x_2)| \leq \epsilon.$$

Recall the following theorem ([3] VIII 6.5).

Theorem 0.1 (Wiener's Tauberian theorem). *Let $K_1 \in L^1(\mathbb{R})$ and $f \in L^\infty(\mathbb{R})$.*

(i) *If $\lim_{x \rightarrow \infty} f(x) = A$, then*

$$\lim_{x \rightarrow \infty} \int_{\mathbb{R}} K_1(x-y)f(y)dy = A \int_{\mathbb{R}} K_1(x)dx.$$

(ii) *Suppose the Fourier transform*

$$\hat{K}_1(\xi) = \int_{\mathbb{R}} K_1(x)e^{i\xi x}dx$$

of K_1 has the property $\hat{K}_1(\xi) \neq 0$ for all $\xi \in \mathbb{R}$ and suppose

$$\lim_{x \rightarrow \infty} \int_{\mathbb{R}} K_1(x-y)f(y)dy = A \int_{\mathbb{R}} K_1(x)dx.$$

Then

$$\lim_{x \rightarrow \infty} \int_{\mathbb{R}} K_2(x-y)f(y)dy = A \int_{\mathbb{R}} K_2(x)dx$$

for all $K_2 \in L^1(\mathbb{R})$. Suppose furthermore that f is weakly oscillating at ∞ . Then we have $\lim_{x \rightarrow \infty} f(x) = A$.

We quickly recall a proof of (ii). Let

$$I = \{K \in L^1(\mathbb{R}) \mid \lim_{x \rightarrow \infty} \int_{\mathbb{R}} K(x-y)f(y)dy = A \int_{\mathbb{R}} K(x)dx\}.$$

Then I is a closed linear subspace of $L^1(\mathbb{R})$. If $K \in I$, then for any $y \in \mathbb{R}$, the translation K_y of K defined by $K_y(x) = K(x-y)$ lies in I . This implies that I is a closed ideal of the Banach

algebra $L^1(\mathbb{R})$. Since $\hat{K}_1(\xi) \neq 0$ for all ξ , by a theorem of Wiener ([3] VIII 6.3), for any $g \in L^1(\mathbb{R})$ such that \hat{g} has compact support, there exists $g_1 \in L^1(\mathbb{R})$ such that $\hat{g} = \hat{g}_1 \hat{K}_1$, which implies that $g = g_1 * K_1$. So the closure of the ideal generated by K_1 is $L^1(\mathbb{R})$. We have $K_1 \in I$, so we have $I = L^1(\mathbb{R})$. Hence for any $K_2 \in L^1(\mathbb{R})$, we have

$$\lim_{x \rightarrow \infty} \int_{\mathbb{R}} K_2(x-y)f(y)dy = A \int_{\mathbb{R}} K_2(x)dx.$$

For any $h > 0$, taking

$$K_2(x) = \begin{cases} \frac{1}{h} & \text{if } x \in [0, h], \\ 0 & \text{if } x \notin [0, h], \end{cases}$$

we get

$$\lim_{x \rightarrow \infty} \frac{1}{h} \int_{x-h}^x f(y)dy = A.$$

If f is weakly oscillating at ∞ , this implies that $\lim_{x \rightarrow \infty} f(x) = A$.

In this paper, we study an analogue of the above result for ℓ -adic sheaves on the affine line. Throughout this paper, p is a prime number, k is an algebraically closed field of characteristic p , \mathbb{F}_p is the finite field with p elements contained in k , ℓ is a prime number distinct from p , and $\psi : \mathbb{F}_p \rightarrow \overline{\mathbb{Q}}_\ell^*$ is a fixed nontrivial additive character. Let $\mathbb{A} = \text{Spec } k[x]$ be the affine line. The Artin-Schreier morphism

$$\wp : \mathbb{A} \rightarrow \mathbb{A}$$

corresponding to the k -algebra homomorphism

$$k[t] \rightarrow k[t], \quad t \mapsto t^p - t$$

is a finite Galois étale covering space, and it defines an \mathbb{F}_p -torsor

$$0 \rightarrow \mathbb{F}_p \rightarrow \mathbb{A} \xrightarrow{\wp} \mathbb{A} \rightarrow 0.$$

Pushing-forward this torsor by ψ^{-1} , we get a lisse $\overline{\mathbb{Q}}_\ell$ -sheaf \mathcal{L}_ψ of rank 1 on \mathbb{A} . Let $\mathbb{A}' = \text{Spec } k[x']$ be another copy of the affine line, let

$$\pi : \mathbb{A} \times_k \mathbb{A}' \rightarrow \mathbb{A}, \quad \pi' : \mathbb{A} \times_k \mathbb{A}' \rightarrow \mathbb{A}'$$

be the projections, and let $\mathcal{L}_\psi(xx')$ be the inverse image of \mathcal{L}_ψ under the k -morphism

$$\mathbb{A} \times_k \mathbb{A}' \rightarrow \mathbb{A}, \quad (x, x') \mapsto xx'$$

corresponding to the k -algebra homomorphism

$$k[t] \rightarrow k[x, x'], \quad t \mapsto xx'.$$

For any object K in the triangulated category $D_c^b(\mathbb{A}, \overline{\mathbb{Q}}_\ell)$ defined in [2] 1.1, the Fourier transform $\mathcal{F}(K) \in \text{ob } D_c^b(\mathbb{A}, \overline{\mathbb{Q}}_\ell)$ of K is defined to be

$$\mathcal{F}(K) = R\pi'_!(\pi^* K \otimes \mathcal{L}_\psi(xx'))[1].$$

Let

$$s : \mathbb{A} \times_k \mathbb{A} \rightarrow \mathbb{A}, \quad (x, y) \mapsto x + y$$

be the k -morphism corresponding to the k -algebra homomorphism

$$k[t] \rightarrow k[x, y], \quad t \mapsto x + y,$$

and let

$$p_1, p_2 : \mathbb{A} \times_k \mathbb{A} \rightarrow \mathbb{A}$$

be the projections. For any $K_1, K_2 \in \text{ob } D_c^b(\mathbb{A}, \overline{\mathbb{Q}}_\ell)$, define their convolution $K_1 * K_2 \in \text{ob } D_c^b(\mathbb{A}, \overline{\mathbb{Q}}_\ell)$ to be

$$K_1 * K_2 = Rs_!(p_1^* K_1 \otimes p_2^* K_2).$$

Let $F \in D_c^b(\mathbb{A}, \overline{\mathbb{Q}}_\ell)$. We say F is a perverse sheaf (confer [1]) if $\mathcal{H}^0(F)$ has finite support, $\mathcal{H}^{-1}(K)$ has no sections with finite support, and $\mathcal{H}^i(K) = 0$ for $i \neq 0, 1$. The Fourier transform of a perverse sheaf on \mathbb{A} is a perverse sheaf on \mathbb{A}' .

Let $\mathbb{P} = \mathbb{A} \cup \{\infty\}$ and $\mathbb{P}' = \mathbb{A}' \cup \{\infty'\}$ be the smooth compactifications of \mathbb{A} and \mathbb{A}' , respectively. They are projective lines. For any Zariski closed point x (resp. x') in \mathbb{P} (resp. \mathbb{P}'), let η_x (resp. $\eta_{x'}$) be the generic point of the henselization of \mathbb{P} (resp. \mathbb{P}') at x (resp. x'), and let $\bar{\eta}_x$ (resp. $\bar{\eta}_{x'}$) be a geometric point above η_x (resp. $\eta_{x'}$). On $\text{Gal}(\bar{\eta}_x/\eta_x)$ (resp. $\text{Gal}(\bar{\eta}_{x'}/\eta_{x'})$), we have a filtration by ramification subgroups in upper numbering. We can use this filtration to define the breaks of $\overline{\mathbb{Q}}_\ell$ -representations of $\text{Gal}(\bar{\eta}_x/\eta_x)$ (resp. $\text{Gal}(\bar{\eta}_{x'}/\eta_{x'})$). For any perverse sheaf F on \mathbb{A} , $\mathcal{H}^{-1}(F)_{\bar{\eta}_x}$ is a $\overline{\mathbb{Q}}_\ell$ -representation of $\text{Gal}(\bar{\eta}_x/\eta_x)$. Confer [5] for the definition of the local Fourier transform $\mathcal{F}^{(x, x')}$.

Theorem 0.2 (Tauberian theorem). *Let $K \in \text{ob } D_c^b(\mathbb{A}, \overline{\mathbb{Q}}_\ell)$ be a perverse sheaf on \mathbb{A} . Suppose the Fourier transform $\mathcal{F}(K)$ is of the form $L[1]$ for some lisse $\overline{\mathbb{Q}}_\ell$ -sheaf L on \mathbb{A}' . Let M, N be lisse $\overline{\mathbb{Q}}_\ell$ -sheaves on \mathbb{A} . Then $K * (M[1])$ and $K * (N[1])$ are perverse.*

(i) *If $M_{\bar{\eta}_\infty} \cong N_{\bar{\eta}_\infty}$, then $\mathcal{H}^{-1}(K * (M[1]))_{\bar{\eta}_\infty} \cong \mathcal{H}^{-1}(K * (N[1]))_{\bar{\eta}_\infty}$.*

(ii) *Suppose L has rank 1, and all the breaks of $L_{\bar{\eta}_\infty} \otimes \mathcal{F}^{(\infty, \infty')}(M_{\bar{\eta}_\infty})$ and $L_{\bar{\eta}_\infty} \otimes \mathcal{F}^{(\infty, \infty')}(N_{\bar{\eta}_\infty})$ lie in $(1, \infty)$. If $\mathcal{H}^{-1}(K * (M[1]))_{\bar{\eta}_\infty} \cong \mathcal{H}^{-1}(K * (N[1]))_{\bar{\eta}_\infty}$, then $M_{\bar{\eta}_\infty} \cong N_{\bar{\eta}_\infty}$.*

Remark 0.3. In Wiener's Tauberian Theorem 0.1, we have $K_1 \in L^1(\mathbb{R})$. This implies that \hat{K}_1 is a uniformly continuous function on \mathbb{R} . This corresponds to the condition in Theorem 0.2 that $\mathcal{F}(K)$ is of the form $L[1]$ for a lisse sheaf L on \mathbb{A}' . There are many perverse sheaves K on \mathbb{A} satisfying this condition. For example, we can start with a lisse sheaf L on \mathbb{A}' , and then take $K = a_* \mathcal{F}'(L[1])(1)$, where \mathcal{F}' is the Fourier transform operator defined as above but interchanging the roles of \mathbb{A} and \mathbb{A}' , $a : \mathbb{A} \rightarrow \mathbb{A}$ is the k -morphism corresponding to the k -algebra homomorphism

$$k[x] \rightarrow k[x], \quad x \mapsto -x,$$

and (1) denotes the Tate twist.

Remark 0.4. As one can see from the proof of Wiener's Tauberian Theorem 0.1, the condition $\hat{K}_1(\xi) \neq 0$ for all ξ ensures that for any $g \in L^1(\mathbb{R})$ such that \hat{g} has compact support, there exists $g_1 \in L^1(\mathbb{R})$ such that $\hat{g} = \hat{g}_1 \hat{K}_1$ and $g = g_1 * K_1$. So the closure of the ideal generated by K_1 in $L^1(\mathbb{R})$. This corresponds to the condition in Theorem 0.2 that $\mathcal{F}(K) = L[1]$ for a lisse sheaf L of rank 1 on \mathbb{A}' . Indeed, for any $G \in \text{ob } D_c^b(\mathbb{A}, \overline{\mathbb{Q}}_\ell)$, we have

$$\begin{aligned} \mathcal{F}(G) &\cong (\mathcal{F}(G) \otimes L^{-1}) \otimes L \\ &\cong (\mathcal{F}(G) \otimes L^{-1}[-1]) \otimes \mathcal{F}(K). \end{aligned}$$

It follows that

$$G \cong G_1 * K,$$

where $G_1 = a_* \mathcal{F}'(\mathcal{F}(G) \otimes L^{-1})(1)$.

Remark 0.5. It is interesting to find a Tauberian theorem in the case where k is of characteristic 0. In this case, the Fourier transform is not available. We need to find a convenient condition on K which ensures that for any $G \in \text{ob } D_c^b(\mathbb{A}, \overline{\mathbb{Q}}_\ell)$, there exists $G_1 \in \text{ob } D_c^b(\mathbb{A}, \overline{\mathbb{Q}}_\ell)$ such that $G \cong G_1 * K$.

By [4] Theorem II 8.1, the condition $\hat{K}(\xi) \neq 0$ for all ξ in Wiener's Theorem 0.1 is equivalent to the condition that if $K * f = 0$ for some $f \in L^\infty(\mathbb{R})$, then we have $f = 0$. So to obtain a Tauberian theorem for ℓ -adic sheaves, we may try to find a condition on a perverse sheaf K on \mathbb{A} which ensures that for any $G \in \text{ob } D_c^b(\mathbb{A}, \overline{\mathbb{Q}}_\ell)$ such that $K * G = 0$, we have $G = 0$.

1 Proof of the Theorem

Keep the notations in the introduction. Denote by

$$\bar{\pi} : \mathbb{P} \times_k \mathbb{P}' \rightarrow \mathbb{P}, \quad \bar{\pi}' : \mathbb{P} \times_k \mathbb{P}' \rightarrow \mathbb{P}'$$

the projections, by $\alpha : \mathbb{A} \hookrightarrow \mathbb{P}$ and $\alpha' : \mathbb{A}' \hookrightarrow \mathbb{P}'$ the immersions, and by $\overline{\mathcal{L}}_\psi(xx')$ the sheaf $(\alpha \times \alpha')^* \mathcal{L}_\psi(xx')$ on $\mathbb{P} \times_k \mathbb{P}'$. For any $\overline{\mathbb{Q}}_\ell$ -representation V of $\text{Gal}(\bar{\eta}_x/\eta_x)$ or $\text{Gal}(\bar{\eta}_{x'}/\eta_{x'})$ and any interval (a, b) in \mathbb{R} , denote by $V^{(a,b)}$ the largest subspace of V with breaks lying in (a, b) .

Lemma 1.1. *Let L, U and V be $\overline{\mathbb{Q}}_\ell$ -representations of $\text{Gal}(\bar{\eta}_x/\eta_x)$. Suppose either $L^{(1,\infty)} = 0$ or $U^{[0,1]} = V^{[0,1]} = 0$.*

(i) *If $U^{(1,\infty)} \cong V^{(1,\infty)}$, then $(L \otimes U)^{(1,\infty)} \cong (L \otimes V)^{(1,\infty)}$.*

(ii) *Suppose furthermore that L has rank 1, and all the breaks of $L \otimes U^{(1,\infty)}$ and $L \otimes V^{(1,\infty)}$ lie in $(1, \infty)$. If $(L \otimes U)^{(1,\infty)} \cong (L \otimes V)^{(1,\infty)}$, then $U^{(1,\infty)} \cong V^{(1,\infty)}$.*

Proof. We have decompositions

$$L \cong L^{[0,1]} \bigoplus L^{(1,\infty)}, \quad U \cong U^{[0,1]} \bigoplus U^{(1,\infty)}.$$

It follows that

$$L \otimes U \cong (L^{[0,1]} \otimes U^{[0,1]}) \bigoplus (L^{(1,\infty)} \otimes U^{[0,1]}) \bigoplus (L \otimes U^{(1,\infty)}).$$

Note that the breaks of $L^{[0,1]} \otimes U^{[0,1]}$ lie in $[0, 1]$, and the breaks of $L^{(1,\infty)} \otimes U^{[0,1]}$ lies in $(1, \infty)$.

It follows that

$$(L \otimes U)^{(1,\infty)} \cong (L^{(1,\infty)} \otimes U^{[0,1]}) \bigoplus (L \otimes U^{(1,\infty)})^{(1,\infty)}.$$

Since either $L^{(1,\infty)} = 0$ or $U^{[0,1]} = 0$, we have

$$(L \otimes U)^{(1,\infty)} \cong (L \otimes U^{(1,\infty)})^{(1,\infty)}.$$

We have a similar equation for V . Our assertion follows immediately. \square

Lemma 1.2. *Let H be a perverse sheaf on \mathbb{A} and let $S \subset \mathbb{A}$ be the set of those closed points s in \mathbb{A} such that either $\mathcal{H}^0(H)_{\bar{s}} \neq 0$ or $\mathcal{H}^{-1}(H)$ is not a lisse sheaf near s . Then we have*

$$\begin{aligned} \left(\mathcal{H}^{-1}(\mathcal{F}(H))_{\bar{\eta}_{\infty'}} \right)^{(1, \infty)} &\cong \mathcal{F}^{(\infty, \infty')}(\mathcal{H}^{-1}(H)_{\bar{\eta}_{\infty}}), \\ \left(\mathcal{H}^{-1}(\mathcal{F}(H))_{\bar{\eta}_{\infty'}} \right)^{[0, 1]} &\cong \bigoplus_{s \in S} R^0 \Phi_{\bar{\eta}_{\infty'}}(\bar{\pi}^* \alpha_! H \otimes \overline{\mathcal{L}}_{\psi}(xx'))_{(s, \infty')}. \end{aligned}$$

Proof. Let $j : \mathbb{A} - S \rightarrow \mathbb{A}$ be the open immersion, and let Δ be the mapping cone of the canonical morphism $j_! j^* H \rightarrow H$. Then Δ has finite support. Hence $\mathcal{H}^i(\mathcal{F}(\Delta))_{\bar{\eta}_{\infty'}}$ are extensions of $\mathcal{L}_{\psi}(ax')|_{\bar{\eta}_{\infty'}}$ for some $a \in k$. In particular, they have no subspace with breaks lying in $(1, \infty)$. We have a distinguished triangle

$$\mathcal{F}(j_! j^* H) \rightarrow \mathcal{F}(H) \rightarrow \mathcal{F}(\Delta) \rightarrow .$$

It follows that

$$\left(\mathcal{H}^{-1}(\mathcal{F}(H))_{\bar{\eta}_{\infty'}} \right)^{(1, \infty)} \cong \left(\mathcal{H}^{-1}(\mathcal{F}(j_! j^* H))_{\bar{\eta}_{\infty'}} \right)^{(1, \infty)}.$$

By [5] 2.3.3.1, we have

$$\mathcal{H}^{-1}(\mathcal{F}(j_! j^* H))_{\bar{\eta}_{\infty'}} \cong \bigoplus_{s \in S} \mathcal{F}^{(s, \infty')}(\mathcal{H}^{-1}(H)_{\bar{\eta}_s}) \bigoplus \mathcal{F}^{(\infty, \infty')}(\mathcal{H}^{-1}(H)_{\bar{\eta}_{\infty}}). \quad (1)$$

We have

$$\mathcal{F}^{(s, \infty')}(\mathcal{H}^{-1}(H)_{\bar{\eta}_s}) \cong \mathcal{F}^{(0, \infty')}(\mathcal{H}^{-1}(H)_{\bar{\eta}_s}) \otimes \mathcal{L}_{\psi}(sx')|_{\bar{\eta}_{\infty'}}.$$

So by [5] 2.4.3 (i) (b), $\mathcal{F}^{(s, \infty')}(\mathcal{H}^{-1}(H)_{\bar{\eta}_s})$ has breaks lying in $[0, 1]$. By [5] 2.4.3 (iii) (b), $\mathcal{F}^{(\infty, \infty')}(\mathcal{H}^{-1}(H)_{\bar{\eta}_{\infty}})$ has breaks lying in $(1, \infty)$. Taking the part with breaks lying in $(1, \infty)$ on both sides of the equation (1), we get the first equation in the lemma. By [5] 2.3.3.1, we have

$$\mathcal{H}^{-1}(\mathcal{F}(H))_{\bar{\eta}_{\infty'}} \cong \bigoplus_{s \in S} R^0 \Phi_{\bar{\eta}_{\infty'}}(\bar{\pi}^* \alpha_! H \otimes \overline{\mathcal{L}}_{\psi}(xx'))_{(s, \infty')} \bigoplus \mathcal{F}^{(\infty, \infty')}(\mathcal{H}^{-1}(H)_{\bar{\eta}_{\infty}}). \quad (2)$$

Taking the part with breaks lying in $[0, 1]$ on both sides of the equation (2), we get the second equation in the lemma. \square

The following proposition apparently looks more general than Theorem 0.2.

Proposition 1.3. *Let $K \in \text{ob } D_c^b(\mathbb{A}, \overline{\mathbb{Q}}_l)$ be a perverse sheaf on \mathbb{A} . Suppose the Fourier transform $\mathcal{F}(K)$ is of the form $L[1]$ for some lisse $\overline{\mathbb{Q}}_l$ -sheaf L on \mathbb{A}' . Let $F, G \in \text{ob } D_c^b(\mathbb{A}, \overline{\mathbb{Q}}_l)$ be perverse sheaves on \mathbb{A} . Then $K * F$ and $K * G$ are perverse. Suppose furthermore either*

$$\mathcal{H}^{-1}(\mathcal{F}(F))_{\bar{\eta}_{\infty'}}^{[0, 1]} = \mathcal{H}^{-1}(\mathcal{F}(G))_{\bar{\eta}_{\infty'}}^{[0, 1]} = 0,$$

or

$$L_{\bar{\eta}_{\infty}'}^{(1,\infty)} = 0.$$

(i) If $\mathcal{H}^{-1}(F)_{\bar{\eta}_{\infty}} \cong \mathcal{H}^{-1}(G)_{\bar{\eta}_{\infty}}$, then $\mathcal{H}^{-1}(K * F)_{\bar{\eta}_{\infty}} \cong \mathcal{H}^{-1}(K * G)_{\bar{\eta}_{\infty}}$.

(ii) Suppose L has rank 1, and all the breaks of

$$L_{\bar{\eta}_{\infty}'} \otimes \mathcal{H}^{-1}(\mathcal{F}(F))_{\bar{\eta}_{\infty}'}^{(1,\infty)} \text{ and } L_{\bar{\eta}_{\infty}'} \otimes \mathcal{H}^{-1}(\mathcal{F}(G)_{\bar{\eta}_{\infty}'})^{(1,\infty)}$$

lie in $(1, \infty)$. If $\mathcal{H}^{-1}(K * F)_{\bar{\eta}_{\infty}} \cong \mathcal{H}^{-1}(K * G)_{\bar{\eta}_{\infty}}$, then $\mathcal{H}^{-1}(F)_{\bar{\eta}_{\infty}} \cong \mathcal{H}^{-1}(G)_{\bar{\eta}_{\infty}}$.

Proof. Denote the Fourier transforms of K and F by \hat{K} and \hat{F} , respectively. Let $a : \mathbb{A} \rightarrow \mathbb{A}$ be the k -morphism corresponding to the k -algebra homomorphism

$$k[x] \rightarrow k[x], \quad x \mapsto -x.$$

By [5] 1.2.2.1 and 1.2.2.7, we have

$$\begin{aligned} K * F &\cong a_* \mathcal{F}' \mathcal{F}(K * F)(1) \\ &\cong a_* \mathcal{F}'(\mathcal{F}(K) \otimes \mathcal{F}(F))[-1](1) \\ &\cong a_* \mathcal{F}'(L \otimes \mathcal{F}(F))(1). \end{aligned}$$

So by [5] 1.3.2.3, $K * F$ is perverse. Let $S' \subset \mathbb{A}'$ be the set of those closed points s' in \mathbb{A}' such that either $\mathcal{H}^0(\mathcal{F}(F))_{\bar{s}'} \neq 0$ or $\mathcal{H}^{-1}(\mathcal{F}(F))$ is not a lisse sheaf near s' . By [5] 2.3.3.1, we have

$$\begin{aligned} \mathcal{H}^{-1}(\mathcal{F}'(L \otimes \mathcal{F}(F)))_{\bar{\eta}_{\infty}} &\cong \bigoplus_{s' \in S'} R^0 \Phi_{\bar{\eta}_{\infty}} \left(\bar{\pi}'^* \alpha'_! (L \otimes \mathcal{F}(F)) \otimes \overline{\mathcal{L}}_{\psi}(xx') \right)_{(\infty, s')} \\ &\quad \oplus \mathcal{F}^{(\infty', \infty)}(L_{\bar{\eta}_{\infty}'} \otimes \mathcal{H}^{-1}(\mathcal{F}(F))_{\bar{\eta}_{\infty}'}). \end{aligned}$$

Since L is lisse on \mathbb{A}' , we have

$$R^0 \Phi_{\bar{\eta}_{\infty}} \left(\bar{\pi}'^* \alpha'_! (L \otimes \mathcal{F}(F)) \otimes \overline{\mathcal{L}}_{\psi}(xx') \right)_{(\infty, s')} \cong L_{\bar{s}'} \otimes R^0 \Phi_{\bar{\eta}_{\infty}} (\bar{\pi}'^* \alpha'_! \mathcal{F}(F) \otimes \overline{\mathcal{L}}_{\psi}(xx'))_{(\infty, s')}.$$

Denote also by a the morphism $\eta_{\infty} \rightarrow \eta_{\infty}$ induced by a . We have

$$\begin{aligned} \mathcal{H}^{-1}(K * F)_{\bar{\eta}_{\infty}} &\cong a_* \left(\bigoplus_{s' \in S'} L_{\bar{s}'} \otimes R^0 \Phi_{\bar{\eta}_{\infty}} (\bar{\pi}'^* \alpha'_! \mathcal{F}(F) \otimes \overline{\mathcal{L}}_{\psi}(xx'))_{(\infty, s')} \right. \\ &\quad \left. \oplus \mathcal{F}^{(\infty', \infty)}(L_{\bar{\eta}_{\infty}'} \otimes \mathcal{H}^{-1}(\mathcal{F}(F))_{\bar{\eta}_{\infty}'}) \right)(1). \end{aligned}$$

By Lemma 1.2, we have

$$\begin{aligned} \mathcal{H}^{-1}(K * F)_{\bar{\eta}_{\infty}}^{(1,\infty)} &\cong a_* (\mathcal{F}^{(\infty', \infty)}(L_{\bar{\eta}_{\infty}'} \otimes \mathcal{H}^{-1}(\mathcal{F}(F))_{\bar{\eta}_{\infty}'}))(1), \\ \mathcal{H}^{-1}(K * F)_{\bar{\eta}_{\infty}}^{[0,1]} &\cong a_* \left(\bigoplus_{s' \in S'} L_{\bar{s}'} \otimes R^0 \Phi_{\bar{\eta}_{\infty}} (\bar{\pi}'^* \alpha'_! \mathcal{F}(F) \otimes \overline{\mathcal{L}}_{\psi}(xx'))_{(\infty, s')} \right)(1). \end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathcal{H}^{-1}(F)_{\bar{\eta}_\infty}^{(1,\infty)} &\cong a_*\left(\mathcal{F}^{(\infty',\infty)}(\mathcal{H}^{-1}(\mathcal{F}(F))_{\bar{\eta}_{\infty'}})\right)(1), \\ \mathcal{H}^{-1}(F)_{\bar{\eta}_\infty}^{[0,1]} &\cong a_*\left(\bigoplus_{s' \in S'} R^0\Phi_{\bar{\eta}_\infty}(\bar{\pi}'^*\alpha'_!\mathcal{F}(F) \otimes \overline{\mathcal{L}}_\psi(xx'))_{(\infty,s')}\right)(1).\end{aligned}$$

Let $T' \subset \mathbb{A}'$ be the set of those closed points s' in \mathbb{A}' such that either $\mathcal{H}^0(\mathcal{F}(G))_{\bar{s}'} \neq 0$ or $\mathcal{H}^{-1}(\mathcal{F}(G))$ is not a lisse sheaf near s' . We have similar equations if we replace F by G and S' by T' .

Suppose $\mathcal{H}^{-1}(F)_{\bar{\eta}_\infty} \cong \mathcal{H}^{-1}(G)_{\bar{\eta}_\infty}$. From

$$\mathcal{H}^{-1}(F)_{\bar{\eta}_\infty}^{[0,1]} \cong \mathcal{H}^{-1}(G)_{\bar{\eta}_\infty}^{[0,1]}. \quad (3)$$

we get

$$\begin{aligned}&a_*\left(\bigoplus_{s' \in S'} R^0\Phi_{\bar{\eta}_\infty}(\bar{\pi}'^*\alpha'_!\mathcal{F}(F) \otimes \overline{\mathcal{L}}_\psi(xx'))_{(\infty,s')}\right)(1) \\ &\cong a_*\left(\bigoplus_{s' \in T'} R^0\Phi_{\bar{\eta}_\infty}(\bar{\pi}'^*\alpha'_!\mathcal{F}(G) \otimes \overline{\mathcal{L}}_\psi(xx'))_{(\infty,s')}\right)(1).\end{aligned} \quad (4)$$

Since L is lisse on \mathbb{A}' , it follows that

$$\begin{aligned}&a_*\left(\bigoplus_{s' \in S'} L_{\bar{s}'} \otimes R^0\Phi_{\bar{\eta}_\infty}(\bar{\pi}'^*\alpha'_!\mathcal{F}(F) \otimes \overline{\mathcal{L}}_\psi(xx'))_{(\infty,s')}\right)(1) \\ &\cong a_*\left(\bigoplus_{s' \in T'} L_{\bar{s}'} \otimes R^0\Phi_{\bar{\eta}_\infty}(\bar{\pi}'^*\alpha'_!\mathcal{F}(G) \otimes \overline{\mathcal{L}}_\psi(xx'))_{(\infty,s')}\right)(1).\end{aligned} \quad (5)$$

that is,

$$\mathcal{H}^{-1}(K * F)_{\bar{\eta}_\infty}^{[0,1]} \cong \mathcal{H}^{-1}(K * G)_{\bar{\eta}_\infty}^{[0,1]}. \quad (6)$$

From

$$\mathcal{H}^{-1}(F)_{\bar{\eta}_\infty}^{(1,\infty)} \cong \mathcal{H}^{-1}(G)_{\bar{\eta}_\infty}^{(1,\infty)}. \quad (7)$$

we get

$$a_*\left(\mathcal{F}^{(\infty',\infty)}(\mathcal{H}^{-1}(\mathcal{F}(F))_{\bar{\eta}_{\infty'}})\right)(1) \cong a_*\left(\mathcal{F}^{(\infty',\infty)}(\mathcal{H}^{-1}(\mathcal{F}(G))_{\bar{\eta}_{\infty'}})\right)(1). \quad (8)$$

So we have

$$\mathcal{F}^{(\infty',\infty)}(\mathcal{H}^{-1}(\mathcal{F}(F))_{\bar{\eta}_{\infty'}}) \cong \mathcal{F}^{(\infty',\infty)}(\mathcal{H}^{-1}(\mathcal{F}(G))_{\bar{\eta}_{\infty'}}). \quad (9)$$

This is equivalent to

$$\mathcal{H}^{-1}(\mathcal{F}(F))_{\bar{\eta}_{\infty'}}^{(1,\infty)} \cong \mathcal{H}^{-1}(\mathcal{F}(G))_{\bar{\eta}_{\infty'}}^{(1,\infty)} \quad (10)$$

by [5] 2.4.3 (iii) (b) and (c). By Lemma 1.1, we have

$$(L_{\bar{\eta}_{\infty'}} \otimes \mathcal{H}^{-1}(\mathcal{F}(F))_{\bar{\eta}_{\infty'}})^{(1,\infty)} \cong (L_{\bar{\eta}_{\infty'}} \otimes \mathcal{H}^{-1}(\mathcal{F}(G))_{\bar{\eta}_{\infty'}})^{(1,\infty)}. \quad (11)$$

Hence

$$\mathcal{F}^{(\infty',\infty)}(L_{\bar{\eta}_{\infty'}} \otimes \mathcal{H}^{-1}(\mathcal{F}(F))_{\bar{\eta}_{\infty'}}) \cong \mathcal{F}^{(\infty',\infty)}(L_{\bar{\eta}_{\infty'}} \otimes \mathcal{H}^{-1}(\mathcal{F}(G))_{\bar{\eta}_{\infty'}}). \quad (12)$$

So we have

$$\mathcal{H}^{-1}(K * F)_{\bar{\eta}_{\infty}}^{(1,\infty)} \cong \mathcal{H}^{-1}(K * G)_{\bar{\eta}_{\infty}}^{(1,\infty)}. \quad (13)$$

By equations (6) and (13), we have

$$\mathcal{H}^{-1}(K * F)_{\bar{\eta}_{\infty}} \cong \mathcal{H}^{-1}(K * G)_{\bar{\eta}_{\infty}}.$$

The above argument can be reversed. We have the following implications for the above equations:

$$\begin{aligned} (3) &\Leftrightarrow (4) \Rightarrow (5) \Leftrightarrow (6), \\ (7) &\Leftrightarrow (8) \Leftrightarrow (9) \Leftrightarrow (10) \Rightarrow (11) \Leftrightarrow (12) \Leftrightarrow (13). \end{aligned}$$

Suppose L has rank 1, then we have $(5) \Rightarrow (4)$. Suppose furthermore that all the breaks of

$$L_{\bar{\eta}_{\infty'}} \otimes \mathcal{H}^{-1}(\mathcal{F}(F))_{\bar{\eta}_{\infty'}}^{(1,\infty)} \text{ and } L_{\bar{\eta}_{\infty'}} \otimes \mathcal{H}^{-1}(\mathcal{F}(G))_{\bar{\eta}_{\infty'}}^{(1,\infty)}$$

lie in $(1, \infty)$. Then by Lemma 1.1, we have $(11) \Rightarrow (10)$. If we have $\mathcal{H}^{-1}(K * F)_{\bar{\eta}_{\infty}} \cong \mathcal{H}^{-1}(K * G)_{\bar{\eta}_{\infty}}$, then (6) and (13) holds. It follows that (3) and (7) holds. We thus have $\mathcal{H}^{-1}(F)_{\bar{\eta}_{\infty}} \cong \mathcal{H}^{-1}(G)_{\bar{\eta}_{\infty}}$. \square

Proof of Theorem 0.2. Theorem 0.2 follows directly from Proposition 1.3 by taking $F = M[1]$ and $G = N[1]$. Since M and N are lisse, by [5] 2.3.3.1 (iii), we have

$$\mathcal{H}^{-1}(\mathcal{F}(F))_{\bar{\eta}_{\infty'}} \cong \mathcal{F}^{(\infty,\infty')}(M_{\bar{\eta}_{\infty}}).$$

By [5] 2.4.3 (iii) (b), the breaks of $\mathcal{F}^{(\infty,\infty')}(M_{\bar{\eta}_{\infty}})$ lie in $(1, \infty)$. Using this fact, one checks that the conditions of Proposition 1.3 hold. \square

Remark 1.4. Proposition 1.3 is actually not more general than Theorem 0.2. Indeed, if

$$\mathcal{H}^{-1}(\mathcal{F}(F))_{\bar{\eta}_{\infty}'}^{[0,1]} = 0,$$

then $\mathcal{F}'\mathcal{F}(F)$ is lisse on \mathbb{A} by [5] 2.3.1.3 (ii), and hence $F = M[1]$ for some lisse sheaf M on \mathbb{A} .

So if we assume the condition

$$\mathcal{H}^{-1}(\mathcal{F}(F))_{\bar{\eta}_{\infty}'}^{[0,1]} = \mathcal{H}^{-1}(\mathcal{F}(G))_{\bar{\eta}_{\infty}'}^{[0,1]} = 0,$$

then Proposition 1.3 is exactly Theorem 0.2. If $L_{\bar{\eta}_{\infty}'}^{(1,\infty)} = 0$, then by the formula [5] 2.3.1.1 (i)', K is a perverse sheaf with finite support. In this case, Proposition 1.3 can be proved directly.

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